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Quantum Mechanics Presented as Harmonic Analysis

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Abstract

This paper grew out of a desire on the part of its author to be able to explain, for philosophy, the significance of Quantum Mechanics. The traditional formulation, based on Hilbert spaces and operators on them, leaves much to be desired if, for example, one is interested in the theory of physical measurement. (It may fairly be asked what the operation on a state function of partial differentiation really has to do with the actual business of measuring the momentum associated with that state function.)

Of great interest here, of course, are the uncertainty relations. They have done much to still the deterministic belief that nature is, at least theoretically, ultimately controllable. They have conjured up again the venerable questions about form versus substance in the theory of matter. They are, then, as fundamental to philosophy today as any other contemporary issue. And yet their explication is deeply problematical because of the relatively complicated form of the usual presentation of these relations. There will be more on this topic in the second section, together with what I believe to be a 'simplest-possible' reformulation.

The reformulation of Quantum Mechanics which is presented below takes its inspiration from the point of view that measurement (or physical knowledge) requires conservation laws, and that these in turn invariably involve symmetry, and hence groups. Thus, as with the uncertainty relations, a concomitant part of knowledge is an associated *inability* to know which arises out of a symmetry group.

Less than that is accomplished here. However, Quantum Mechanics *is* brought significantly closer to group theory, and ease of interpretation. Further, the techniques are sufficiently simple, mathematically, and in terms of justification of the methods used, that they might well be substituted for the traditional approach for the teaching of this subject to undergraduates.

1. *The Fundamentals of Harmonic Analysis*

Presented here are enough of the definitions and results of Harmonic Analysis to make the approach below understandable to the initiate in this area. This material is readily available in more detail. See, for example, Reiter (1968).

Definition 1.1. A *locally compact abelian group*, G, is an abelian group which has been endowed with a topology (system of open subsets) in such a way as to make the operations of multiplication, and the taking of inverses, continuous. Also, each element of G has a compact neighbourhood.

(The purpose of topology here is, as usual, to make sense of the term 'continuous' in a general context.)

Definition 1.2. The *dual group*, G^*, of G consists of all continuous homomorphisms g^* of G to the unit circle in the complex plane. Multiplication in G^* is defined by

$$(g_1^* g_2^*)(g) = g_1^*(g) g_2^*(g) \qquad \text{for all } g \text{ in } G$$

The topology of G^* is obtained by using the 'compact-open' topology, defined below.

Definition 1.3. The 'compact-open' topology of G^* is obtained by choosing as a sub-basis of open sets all the sets $U(A, P)$, where A is a compact subset of G, P is a simple open arc of the unit circle in the complex plane, and g^* is in $U(A, P)$ if, and only if, $g^*(A)$ lies wholly in P.

With these definitions we have:

Theorem 1.1. G^* is a locally compact abelian group if G is. Further, $(G^*)^*$ is naturally homomorphic to G.

It might be noted, by way of example, that R^n (with the usual topology) has dual group R^n with, again, the usual topology. The integers under addition, with the discrete topology (all subsets are open), have as dual group the group of rotations of the circle, which is compact, and vice versa. Indeed, we have:

Theorem 1.2. If G is compact, then G^* is discrete, and vice versa.

Theorem 1.3. Let $f: G \to C$ be a function from G to the complex numbers, and suppose that f is continuous and has compact support (i.e. vanishes outside some compact subset of G). Then the *Haar Integral*, $\int_G f(g)$, is defined, with all the usually expected properties of an integral. It has the additional property that it is invariant under G-translation. With this requirement, it is unique up to multiplication by a fixed positive scalar.

This sense of integration can sometimes be extended to include some functions which do not have compact support. R^n is a good case in point.

Definition 1.4. Let $f: G \to C$ be, as above, a function from G to the complex numbers. We define the *dual function*, f^*, of f, as a function from G^* to C by

$$f^*(g^*) = \int_G f(g) g^*(g^{-1})$$

In case G is R^n, f^* is simply the Fourier transform of f. Indeed, this definition, and Harmonic Analysis, generalises Fourier Analysis.

Theorem 1.4. (Plancherel.) With a suitable (and available) choice of Haar integral for G^*,

$$\int_G |f(g)|^2 = \int_{G^*} |f^*(g^*)|^2$$

This tells us that if $|f(g)|^2$ can be regarded as a probability distribution on G, by dint of

$$\int_G |f(g)|^2 = 1$$

then $|f^*(g^*)|^2$ can be regarded as a probability distribution on G^* for the same reason.

Theorem 1.5. $\int_{G^*} f^*(g^*)g^*(g) = f(g)$. That is: $(f^*)^* = f$.

Throughout this paper G will stand for a locally compact abelian group, G^* for its dual, and $f(g)$ a function from G to the complexes with L_2 norm 1.

2. *The Uncertainty Relations*

Consider the following theorem, which, it is claimed here, is the ideal expression of the uncertainty relations:

(There have been many variants on the classical Heisenberg formulation. See, for example, Bruijn (1965) and Weyl (1950).

Theorem 2.1. Let f, f^*, G and G^* be as above, and suppose that f and f^* are in L_1 over their respective groups. Then

$$\int_G |f(g)| \int_{G^*} |f^*(g^*)| \geqslant 1$$

Proof.

$$\int_G |f(g)| \int_{G^*} |f^*(g^*)| = \int_G \int_{G^*} |\overline{f(g)} f^*(g^*)|$$

$$= \int_G \int_{G^*} |\overline{f(g)} f^*(g^*) g^*(g)| \qquad \text{(because } |g^*(g)| = 1\text{)}$$

$$\geqslant \left| \int_G \overline{f(g)} \int_{G^*} f^*(g^*) g^*(g) \right|$$

$$= \left| \int_G \overline{f(g)} f(g) \right| \qquad \text{(from Theorem 1.5)}$$

$$= \int_G |f(g)|^2 = 1$$

Before this result can become relevant to physics, it is necessary first to remember that if G is regarded as 'position space', then G^* may be regarded as 'frequency space'. Further, if $|f(g)|^2$ is regarded as a probability distribution over G, then $|f^*(g^*)|^2$ is the associated probability distribution over G^*. Finally, the L_1 norm of $f(g)$ is a decent and adequate measure of the entropy associated with the probability distribution $|f(g)|^2$. That this is so may be seen by observing that

$$\int_G |f(g)|.1$$

then $|f(g)|^2$ cannot have compact support U with measure $\mu(U)$ less than t^2, since otherwise

$$t = \int_G |f(g)| = \int_U |f(g)| \leqslant \left(\int |f|^2\right)^{1/2} \left(\int_U 1\right)^{1/2} = (\mu(U))^{1/2} < t.$$

This result is never invalidated by an absence of functions f for which all of the integrals needed here are defined. Indeed, P. Milnes (private correspondence) has shown that the functions f satisfying

$$f \in L_1 \quad \text{and} \quad L_2 \quad \text{over } G$$

and

$$f^* \in L_1 \quad \text{and} \quad L_2 \quad \text{over } G^*$$

are dense in the set of all continuous functions from G to C which vanish at infinity.

Unfortunately, our description of G^* as 'frequency space' is somewhat unclear from a physical point of view. Indeed, even the notion of G as 'position space' leaves something to be desired if we wish to make some quantitative sense of the term 'position'. We do this for both G and G^* by considering continuous representations of G to R^+, the reals under addition.

Let $\rho : G \to R^+$ be such a representation of G. With respect to ρ we might say that the 'distance from q_1 to g_1' is

$$|\rho(g_1) - \rho(g_2)| = |\rho(g_1 g_2^{-1})|$$

Let K_ρ be the kernel of ρ.

$$K_\rho = \{g \in G \mid \rho(g) = 0\}$$

G/K_ρ is isomorphic to the range of ρ. Also, observe that for any real λ, $(\lambda\rho)(g) = \lambda(\rho(g))$ is again a representation of G to R^+. Finally, note that the mapping

$$g \to e^{i\lambda\rho(g)}$$

is a continuous representation of G to the unit circle, and so is an element of G^*, say $g^*_{\lambda\rho}$.

Now the mapping β defined by $\beta(g^*_{\lambda\rho}) = \lambda$ is a homomorphism from the subgroup of G^* consisting of the $g^*_{\lambda\rho}$ provided that $g^*_{\lambda\rho} \neq 1$ for $\lambda \neq 0$ in R. If this fails to obtain, then there is a positive λ_0 such that

$$e^{i}\lambda_0\rho(g) \equiv 1$$

or,

$$\lambda_0\rho(g) \equiv 0 \ \mathrm{Mod}(2\pi)$$

or,

$$\rho(g) \equiv 0 \ \mathrm{Mod}(2\pi/\lambda_0)$$

and $G/K_\rho = Z$, the integers.

If we assume that this last, exceptional case is not the one we are dealing with then we may interpret G and G^*, relative to ρ, as follows:

$|\rho(g_1 g_2^{-1})|$ is the distance from g_1 to g_2 while,

$\beta(g^*_{\lambda\rho})$ is a physical quantity (momentum in the ρ-direction) which is preserved under G-translation.

In case G/K_ρ *is* Z, we can interpret $\beta(g^*_{\lambda\rho})$ as angle, and $\rho(g)$ as angular momentum. There will be more on this subject in the section 'Generalising Planck's Assumption'.

3. *Differentiation*

Any analogue of quantum mechanics must make some sense of the notion of the derivative of a function. The functions, f, which we deal with here are functions from G to C.

We introduce the following *notation,* soon to be justified as a sense of partial differentation: If $\rho : G^* \to R^+$ is a continuous representation of G^* into the additive group of the reals, and $f: G \to C$ is a function from G to C, we write:

$$\frac{\partial f(g)}{\partial \rho} = i \int_{G^*} f^*(g^*)\rho(g^*)g^*(g) \tag{3.1}$$

The reader may easily verify the fact that if $G = R^n$, $\partial f/\partial\rho$ is simply one of the directional derivatives of f. This result is standard in Fourier analysis, and follows from an integration by parts.

In order to justify regarding the operation $\partial/\partial\rho$ as a differentiation, we offeı the following:

Theorem 3.1. Let ρ and β be continuous representations of G^* to R^+, and let f, h be functions from G to C. Suppose that integrals are defined as needed. Then:

(a) $\dfrac{\partial}{\partial\rho}(f+h)(g) = \dfrac{\partial}{\partial\rho}f(g) + \dfrac{\partial}{\partial\rho}h(g).$

(b) If $\lambda \in C$, $\dfrac{\partial}{\partial\rho}(\lambda f)(g) = \lambda\dfrac{\partial}{\partial\rho}(f)(g).$

(c) $\dfrac{\partial}{\partial\rho}(fh)(g) = h(g)\dfrac{\partial}{\partial\rho}f(g) + f(g)\dfrac{\partial}{\partial\rho}h(g).$

(d) If $\rho+\beta$ be defined in the usual way (viz. $(\rho+\beta)(g^*) = \rho(g^*) + \beta(g^*)$) then

$$\frac{\partial}{\rho+\beta}f = \frac{\partial}{\partial\rho}f + \frac{\partial}{\partial\beta}f$$

(e) $\dfrac{\partial}{\partial\rho}\left(\dfrac{\partial}{\partial\beta}f\right)(g) = \dfrac{\partial}{\partial\beta}\left(\dfrac{\partial}{\partial\rho}f\right)(g).$

(f) $\dfrac{\partial}{\partial\rho}\left(\dfrac{\partial}{\partial\rho}f\right)(g) = -\displaystyle\int_{G^*} f^*(g^*)(\rho(g^*))^2 g^*(g).$

(g) If f and h are identical in a neighbourhood of g, then $\frac{\partial}{\partial\rho}f(g) = \frac{\partial}{\partial\rho}h(g)$ (Locality of the derivative.)

(h) $\int_G \frac{\partial}{\partial\rho} f(g) = 0.$

(i) If f is real, then $\frac{\partial}{\partial\rho} f$ is real also.

Proof: (a), (b) and (d) follow immediately. To show (c) we examine

$$\frac{\partial}{\partial\rho}(fh)(g) = i \int_{G^*} (fh)^*(g^*)\rho(g^*)g^*(g)$$
$$= i \int_{G^*} \int_G (fh)(k)g^*(k^{-1})\rho(g^*)g^*(g)$$
$$= i \int_{G^*} \int_G f(k)h(k)g^*(k^{-1}g)\rho(g^*)$$

We use

$$f(k) = i \int_{G^*} f^*(r^*)r^*(k)$$

$$h(k) = i \int_{G^*} h^*(s^*)s^*(k)$$

and

$$\int_G q^*(k) = \int_G k(q^*) = -i\delta\,(q^* = 1^*)$$

$$\frac{\partial}{\partial\rho}(fh)(g) = -i \int_{G^*}\int_{G^*}\int_{G^*}\int_G f^*(r^*)h^*(s^*)r^*(k)s^*(k)g^*(k^{-1}g)\rho(g^*)$$
$$= -i \int_{G^*}\int_{G^*}\int_{G^*} (f^*(r^*)h^*(s^*)g^*(g)\rho(g^*)) \int_G k(r^*s^*g^{*-1})$$
$$= - \int_{G^*}\int_{G^*}\int_{G^*} f^*(r^*)h^*(s^*)g^*(g)\rho(g^*)\delta\,(g^* = r^*s^*)$$
$$= - \int_{G^*}\int_{G^*} f^*(r^*)h^*(s^*)r^*(g)s^*(g)\rho(r^*s^*)$$
$$= - \int_{G^*}\int_{G^*} f^*(r^*)h^*(s^*)r^*(g)s^*(g)(\rho(r^*) + \rho(s^*))$$
$$= - \int_{G^*} f^*(r^*)r^*(g)\rho(r^*) \int_{G^*} h^*(s^*)s^*(g)$$
$$= - \int_{G^*} f^*(r^*)r^*(g) \int_{G^*} h^*(s^*)s^*(g)\rho(s^*)$$
$$= \frac{\partial}{\partial\rho}f(g)h(g) + f(g)\frac{\partial}{\partial\rho}h(g), \qquad \text{as required.}$$

To show

$$\frac{\partial}{\partial\beta}\left(\frac{\partial}{\partial\rho}f\right)(g)=\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\beta}f\right)(g)$$

we need only note that $(\partial/\partial\rho)f(g)$ is the function whose dual function is $i\rho(g^*)f^*(g^*)$. Our result follows from the observation that

$$-\beta(g^*)\rho(g^*)f^*(g^*)=-\rho(g^*)\beta(g^*)f^*(g^*)$$

(f) follows in the same way. To show (g) (locality of the derivative) we proceed by cases:

Case 1. Suppose $f(g)\geqslant 0$ for all g in G, and $f(g)=0$ at g_0. We want to show that $(\partial/\partial\rho)f(g_0)=0$. To do so, let $h(g)=f^{1/2}(g)$. Then

$$\frac{\partial}{\partial\rho}f(g_0)=\frac{\partial}{\partial\rho}h^2(g_0)=2h(g_0)\frac{\partial}{\partial\rho}h(g_0)\qquad(\text{by } c))=0$$

$(\partial/\partial\rho)h(g_0)=0$.

The same result follows easily in case $f(g)\leqslant 0$ for all g.

Case 2. Suppose $f(g)$ is real-valued, and $f(g)=0$ in a neighbourhood of g_0. We want, again, that $(\partial/\partial\rho)f(g_0)=0$. We may write f as f_1+f_2 where

$$\begin{cases}f_1(g)=f(g) & \text{if } f(g)\geqslant 0\\ 0 & \text{otherwise}\end{cases}$$

and

$$\begin{cases}f_2(g)=f(g) & \text{if } f(g)\leqslant 0\\ 0 & \text{otherwise}\end{cases}$$

Then

$$\frac{\partial}{\partial\rho}f(g_0)=\frac{\partial}{\partial\rho}f_1(g_0)+\frac{\partial}{\partial\rho}f_2(g_0)=0$$

by the results of Case 1.

Case 3. If $f:G\to C$ is 0 in a neighbourhood of g_0, then $(\partial/\partial\rho)f(g_0)=0$.

We proceed in a manner similar to the proof of Case 2, writing $f=f_1+if_2$, where f_1 and f_2 are both real-valued. The result follows immediately as in Case 2.

Case 4. If $f:G\to C$ and $h:G\to C$ are identical in a neighbourhood of g_0, then $(\partial/\partial\rho)f(g_0)=(\partial/\partial\rho)h(g_0)$. To see this, we need only consider $f-h$, and apply Case 3.

We now show (i): That if f is real, so too is $(\partial/\partial\rho)f$.

This result hinges on the fact that the Haar integral, taken over an abelian group, is invariant under the replacement of the argument by its inverse. This is easily shown, and will not be proved here, only used. Now:

$$\overline{\frac{\partial}{\partial\rho}f(g)} = -i \int_{G^*}\int_G f(h)\overline{g^*(h^{-1}g)}\rho(g^*) \qquad \text{(because } f(h) \text{ and } \rho(g^*) \text{ are real)}$$

$$= -i \int_{G^*}\int_G f(h)(g^*)^{-1}(h^{-1}g)\rho(g^*)$$

$$= -i \int_{G^*}\int_G f(h)g^*(h^{-1}g)\rho((g^*)^{-1}) \qquad \text{(by the remark above)}$$

$$= +i \int_{G^*}\int_G f(h)g^*(h^{-1}g)\rho(g^*) = \frac{\partial}{\partial\rho}f(g)$$

It only remains to show that

$$\int_G \frac{\partial}{\partial\rho}f(g) = 0$$

We have

$$\int_G \frac{\partial}{\partial\rho}f(g) = \int_G\int_{G^*}\int_G f(h)g^*(gh^{-1})\rho(g^*)$$

$$= \int_G\int_{G^*} f(h)g^*(h^{-1})\rho(g^*)\int_G g^*(g)$$

$$= \int_G\int_{G^*} f(h)g^*(h^{-1})\rho(g^*)\delta(g^* = 1^*)$$

$$= \int_G f(h)1^*(h^{-1})\rho(1^*)$$

$$= \int_G f(h)\,.\,1\,.\,0 = 0$$

This sense of differentiation will prove to be the key in interpreting quantum mechanics in an abelian group context. It will, in particular, allow us to present the Schroedinger equation as an integral equation, derived from an entirely simple and natural principle, which we present in the next section.

4. *The Schroedinger Equation*

It is one thing to have a candidate for a replacement for the conventional uncertainty relations: It is another to show that these can be developed in a context which actually permits the usual and expected calculations of quantum mechanics. That is what is done in this section.

Let us first briefly review the situation for conventional quantum mechanics in its most simple-minded terms. (That is, based on classical physics, without spin or time considerations, and without concern for systems of identical particles.) In this context we may assume that there is given a physical quantity $P(\mathbf{x}, \mathbf{p})$, which is a function of variables $\mathbf{x}$ (in R^n), the position vector, and $\mathbf{p}$ (in $(R^n)^*$), the momentum vector. Elementary quantum mechanics consists in replacing p_i by $ih(\partial/\partial x_i)$ and solving the partial differential equation (called the *Schroedinger equation*)

$$P\left(\mathbf{x}, ih\frac{\partial}{\partial x}\right)f(\mathbf{x}) = P_0 f(\mathbf{x})$$

The eigenvalues P_0 are, in this theory, the observed values which will arise when the physical quantity $P(\mathbf{x}, \mathbf{p})$ is determined experimentally.

We note in passing that this substitutional rule leading to a differential equation is in trouble if either the p_i do not arise in purely integral powers or if the terms x_i and p_i are mixed in multiplication. This last is problematical because, while x_i and p_i commute, x_i and $\partial/\partial x_i$ do not, and the operator $P[\mathbf{x}, ih(\partial/\partial x)]$ may, as a result, be ill-defined. Both of these difficulties disappear in the approach below.

Yet another complaint with this conventional approach arises when one tries to assess the contribution of quantum mechanics to the theory of knowledge, or to the theory of measurement. To wit: The substitution $p_i \to ih(\partial/\partial x_i)$ rests upon grounds which are difficult to generalise, and rather overly technical.

For our purposes here we shall presuppose Planck's assumption which will be interpreted to be the statement that if G is position space, then G^* is momentum space or, equivalently, frequency space. Thus we shall take as physical expressions functions $P(g, g^*)$ from $G \times G^*$ to R. This done, it remains to find the analogue of the Schroedinger equation.

In doing this, it is worth while keeping in mind the physical interpretation of the mathematical devices being used:

G—Each element represents a 'position'.

G^*—Each element represents a 'frequency'.

$f: G \to C$ is a state function.

$\int_{S \subset G} |f(g)|^2$ is the probability that the particle will be found within S.

Suppose, then, that we are given a real-valued function $P(g, g^*)$ from $G \times G^* \to R$ which represents some physical quantity, such as energy. We postulate the following:

Definition 4.1. The 'pure' states, $f(g)$, associated with $P(g, g^*)$ are those which produce local minima (on average) of $P(g, g^*)$ over both G and G^*.

Put more precisely this says that the pure states, $f(g)$, associated with $P(g, g^*)$ are those for which

$$\int_G \int_{G^*} P(g, g^*)\,|f(g)|^2\,|f^*(g^*)|^2$$

is a local minimum, subject to

$$\int_G |f(g)|^2 = 1$$

We then have the following:

Theorem 4.1. If $f(g)$ is a pure state associated with the physical quantity $P(g, g^*)$, then $f(g)$ satisfies the equation

$$f(g) \int_G P(g, g^*)|f^*(g^*)|^2 + \int_G \int_{G^*} P(h, g^*)|f(h)|^2 f^*(g^*) g^*(g) = 2P^0 f(g)$$

$$\text{for all } g \text{ in } G \qquad (4.1)$$

where P^0 is the average value of $P(g, g^*)$ over G and G^* with respect to f and f^*.

Proof. Minimising $\int_G \int_{G^*} P(g, g^*)|f(g)|^2 |f^*(g^*)|^2$ by choice of f subject to $\int_G |f(g)|^2 = 1$ is the same as minimising

$$I(f) = \int_G \int_{G^*} P(g, g^*)|f(g)|^2 |f^*(g^*)|^2 \Big/ \left(\int_G |f(t)|^2 \right)^2$$

subject to no constraints at all.

$$\left(\text{Since} \quad \int_{G^*} |f^*(g^*)|^2 = \int_G |f(g)|^2 \right)$$

The techniques of minimising integrals are well known. We use a standard one here. Let $f(g)$ be replaced by $f(g) + \epsilon\eta(g)$, where ϵ is in R, and $\eta(g)$ is chosen arbitrarily as a function from G to C, subject to the constraint that all the integrals used continue to be defined. We have then $I(f + \epsilon\eta)$. For a minimum we insist that

$$\frac{d}{d\epsilon} I(f + \epsilon\eta) = 0 \qquad \text{at} \qquad \epsilon = 0, \qquad \text{for all choices of } \eta$$

To perform this calculation we first write $I(f)$ as

$$I(f) = \int_G \int_G \int_G \int_{G^*} P(g, g^*) f(g) \overline{f(g)} f(h) \overline{f(k)} g^*(h^{-1}k) / \left(\int_G f(t) \overline{f(t)} \right)^2$$

$I(f + \epsilon\eta)$ then takes the form:

$$\frac{\int_G \int_G \int_G \int_{G^*} P(g, g^*)(f(g) + \epsilon\eta(g))(\overline{f(g)} + \epsilon\overline{\eta(g)})(f(h) + \epsilon\eta(h))(f(k) + \epsilon\overline{\eta(k)}) g^*(h^{-1}k)}{\left(\int_G (f(t) + \epsilon\eta(t))(\overline{f(t)} + \epsilon\overline{\eta(t)}) \right)^2}$$

It follows directly from $(d/d)I(f+\epsilon\eta)|_{\epsilon=0}$ and $\int_G |f(t)|^2 = 1$ that

$$\int_G \int_G \int_G \int_{G^*} P(g, g^*)(\eta(g)\overline{f(g)}f(h)\overline{f(k)} + f(g)\overline{\eta(g)}f(h)\overline{f(k)}$$

$$+ f(g)\overline{f(g)}\eta(h)\overline{f(k)} + f(g)\overline{f(g)}f(h)\overline{\eta(k)})g^*(h^{-1}k)$$

$$= 2\int_G \int_{G^*} P(g, g^*)|f(g)|^2|f^*(g^*)|^2 \cdot \int_G (\eta(t)\overline{f(t)} + \overline{\eta(t)}f(t))$$

for all η.

In order to employ the arbitrariness of the $\eta(g)$ we rewrite the above so as to involve g as the variable in each occurrence of the η, obtaining,

$$\int_G \int_G \int_G \int_{G^*} P(g, g^*)(\eta(g)\overline{f(g)}f(h)\overline{f(k)} + f(g)\overline{\eta(g)}f(h)\overline{f(k)})g^*(h^{-1}k)$$

$$+ \int_G \int_G \int_G \int_{G^*} P(h, g^*)(f(h)\overline{f(h)}\eta(g)\overline{f(k)})g^*(g^{-1}k)$$

$$+ \int_G \int_G \int_G \int_{G^*} P(h, g^*)(\overline{f(h)}f(h)f(k)\overline{\eta(g)})g^*(k^{-1}g)$$

$$= 2\int_G \int_{G^*} P(h, g^*)|f(h)|^2|f^*(g^*)|^2 \cdot \int_G (\eta(g)\overline{f(g)} + \overline{\eta(g)}f(g)$$

The arbitrariness of $\eta(g)$, together with the independence of $\eta(g)$ and $\overline{\eta}(g)$ gives us

$$f(g)\int_G \int_G \int_{G^*} P(g, g^*)f(h)\overline{f(k)}g^*(h^{-1}k)$$

$$+ \int_G \int_G \int_{G^*} P(h, g^*)|f(h)|^2 f(k)g^*(k^{-1}g)$$

$$= 2\int_G \int_{G^*} P(h, g^*)|f(h)|^2|f^*(g^*)|^2 f(g)$$

But

$$\int_G \int_{G^*} P(h, g^*)|f(h)|^2|f^*(g^*)|^2 = P^0$$

and we have

$$f(g)\int_{G^*} P(g, g^*)|f^*(g^*)|^2 + \int_G \int_{G^*} P(h, g^*)|f(h)|^2 f^*(g^*)g^*(g) = 2P^0 f(g)$$

Corollary 4.1. If $P(g, g^*)$ takes the form $P(g, g^*) = P_1(g) + P_2(g^*)$, then (4.1) takes the form

$$\int_{G^*} P(g, g^*)f^*(g^*)g^*(g) = f(g)P^0 \tag{4.2}$$

Proof.

$$\text{Let } P_1{}^0 = \int_G P_1(g)\,|f(g)|^2 \qquad \text{and} \qquad P_2{}^0 = \int_{G^*} P_2(g^*)\,|f^*(g^*)|^2$$

Then we have $P^0 = P_1{}^0 + P_2{}^0$. Now with $P(g, g^*) = P_1(g) + P_2(g^*)$, (4.1) becomes

$$f(g) \int_{G^*} (P_1(g) + P_2(g^*))\,|f^*(g^*)|^2 + \int_G \int_{G^*} (P_1(h) + P_2(g^*)) f(h)^2 f^*(g^*) g^*(g)$$
$$= 2(P_1{}^0 + P_2{}^0) f(g)$$

or

$$f(g)P_1(g) \int_{G^*} |f^*(g^*)|^2 + f(g) \int_{G^*} P_2(g^*)\,|f^*(g^*)|^2$$
$$+ \int_G \int_{G^*} P_1(h)\,|f(h)|^2 f^*(g^*) g^*(g) + \int_G \int_{G^*} P_2(g^*)\,|f(h)|^2 f^*(g^*) g^*(g)$$
$$= 2(P_1{}^0 + P_2{}^0) f(g)$$

Thus,

$$f(g)P_1(g) + f(g)P_2{}^0 + \int_G P_1(h)\,|f(h)|^2 \int_{G^*} f^*(g^*) g^*(g)$$
$$+ \int_G |f(h)|^2 \int_{G^*} P_2(g^*) f^*(g^*) g^*(g) = 2(P_1{}^0 + P_2{}^0) f(g)$$

and

$$P_1(g) \int_{G^*} f^*(g^*) g^*(g) + f(g)P_2{}^0 + P_1{}^0 f(g) + \int_{G^*} P_2(g^*) f^*(g^*) g^*(g)$$
$$= 2(P_1{}^0 + P_2{}^0) f(g)$$

or

$$P_1(g) \int_{G^*} f^*(g^*) g^*(g) + \int_{G^*} P_2(g^*) g^*(g) = (P_1{}^0 + P_2{}^0) f(g)$$

or

$$\int_{G^*} (P_1(g) + P_2(g^*)) f^*(g^*) g^*(g) = (P_1{}^0 + P_2{}^0) f(g)$$

or

$$\int_{G^*} P(g, g^*) f^*(g^*) g^*(g) = P^0 f(g)$$

as claimed.

The example of a single particle in a force field will illustrate how this formulation works and will show, in that simple instance, how the approach here leads to the 'conventional' Schroedinger equation.

The expression for the energy of such a system is $E(\mathbf{x}, \mathbf{p}) = p_1^2 + p_2^2 + p_3^2 + V(\mathbf{x})$. The group G here is R^3, while G^* is the usual dual $(R^3)^* \cong R^3$. We have three 'natural' representations

$$\rho_i \qquad (i = 1, 2, 3) \qquad \text{of} \qquad (R^3)^*$$

via

$$\rho_i(x_1^*, x_2^*, x_3^*) = x_i^*$$

We have also that

$$p_i = \hbar\rho_i(\mathbf{x}^*) = \hbar x_i^* \qquad (i = 1, 2, 3)$$

Making this substitution, $E(\mathbf{x}, \mathbf{p})$ becomes

$$E(\mathbf{x}, \mathbf{x}^*) = \hbar^2(\rho_1^2(\mathbf{x}^*) + \rho_2^2(\mathbf{x}^*) + \rho_3^2(\mathbf{x}^*)) + V(\mathbf{x})$$

Since $E(\mathbf{x}, \mathbf{x}^*)$ splits into a sum of its $\mathbf{x}$ part and its $\mathbf{x}^*$ part, we may use the equation (4.2) of Corollary 4.1 to obtain, for the state function $f(\mathbf{x})$, the condition

$$\int_{(R^3)^*} [\hbar^2(\rho_1^2(\mathbf{x}^*) + \rho_2^2(\mathbf{x}^*) + \rho_3^2(\mathbf{x}^*)) + V(\mathbf{x})] f^*(\mathbf{x}^*)\mathbf{x}^*(\mathbf{x}) = E_0 f(x)$$

(Here, $\mathbf{x}^*(\mathbf{x}) = \exp[i(x_1^* x_1 + x_2^* x_2 + x_3^* x_3)]$.)

It is at this point in showing that equivalence of the Haar integral formulation and the usual Schroedinger equation that we use the notion of derivative developed in the third section. Our equation becomes

$$\left[-\hbar^2\left(\frac{\partial^2}{\partial\rho_1^2} + \frac{\partial^2}{\partial\rho_2^2} + \frac{\partial^2}{\partial\rho_3^2}\right) + V(\mathbf{x})\right] f(\mathbf{x}) = E_0 f(\mathbf{x})$$

In this instance Fourier analysis shows, without difficulty, that $\partial/\partial\rho_i = \partial/\partial x_i$, and we have, clearly, achieved the usual and accepted differential equation for this system.

In fact, so long as we stay with groups G, which are isomorphic to R^n, and with functions $E(g, g^*)$, which split as $E(g, g^*) = E_1(g) + E_2(g^*)$, it is soon clear that the equation of Corollary 4.1 is equivalent to the differential equation

$$E\left(x_1, \ldots, x_n;\quad ih\frac{\partial}{\partial x_1}, \ldots, ih\frac{\partial}{\partial x_n}\right) f(x_1, \ldots, x_n) = E_0 f(x_1, \ldots, x_n)$$

In the event that these last assumptions should *fail* to hold, then equation (4.1) remains usable, while the usual operator substitution becomes difficult, arbitrary, or impossible.

5. *Generalizing Planck's Assumption*

Quantum mechanics was truly and simply born from the assumptions

$$E = hf$$

and

$$p_i = hf_i$$

where E, p_i, f and f_i are, respectively, the *average* energy, momentum, time-frequency, and length-frequency in the ith direction of an *individual* particle. ('Length-frequency' refers to the number of wave-lengths per unit interval.) It was these assumptions which forced energy and momentum to be regarded as points in the dual of time-position space, rather than simply as measurable observables in time-position space.

It is necessary here for us to have some formulation of these equations for arbitrary locally compact abelian groups, other than R^4. To this end we make the following observations:

(i) The ith component, p_i, of the momentum of a particle is said to be a 'conserved quantity' chiefly because it is invariant under G-translation. Similarly, energy is invariant under G-translation.

(ii) The central and fundamental assumption of Quantum Mechanics is that these *conserved* quantities measured in position-space are proportional to quantities in the dual of position-space.

(iii) The state functions $f : G \to C$ that we have used have not as yet been provided with a thorough physical interpretation. While it is clear enough that

$$\int_{S \subset G} |f(g)|^2$$

may be interpreted either as the probability of finding the particle in S, or as the average density of that particle in S, it is not yet quite clear what the *phase* of $f(g)$ represents physically.

We begin with the following simple observation, of a purely mathematical nature:

If ρ: $G^* \to R^+$ is a continuous homomorphism from G^* to the reals under addition, then $g^* \to e^{i\rho(g^*)}$ is a continuous homomorphism from G^* to the unit circle of the complex plane. Hence this last mapping is an element of $(G^*)^*$, which is homomorphic to G. It follows that there is a g in G such that

$$g^*(g) = e^{i\rho(g^*)} \qquad \text{for all } g^*$$

and that the notation $(\partial/\partial\rho)f$, for functions $f : G \to C$, may be replaced by the equivalent notation $(\partial/\partial g)f$.

The state function $f(g)$ may be written as

$$f(g) = R(g)\, e^{i\theta(g)}, \qquad \text{with } R(g) \geqslant 0$$

$R(g)$ was interpreted in (iii), above. It remains to interpret $\theta(g)$, and we do this as follows:

Choose h in G. Then $(\partial\theta/\partial h)(g)$ is proportional to a conserved physical quantity (momentum, energy, etc.).

$\partial\theta/\partial h$ is not, however, always defined. In order to achieve greater generality we must revert to representations $\rho : G^* \to R^+$, and speak of $\partial\theta/\partial\rho$. A choice of ρ still determines a one-dimensional subgroup, H, of G, as follows:

Let $K^*(\rho)$ be the kernal of ρ in G^*. Then H consists of those h in G for which

$$g^* \in K^*(\rho) \Rightarrow g^*(h) = 1$$

Hence we have:

Definition 5.1. The *average* (generalised) *energy*, E_ρ, in the direction ρ of a system in a group G with state function $f(g)$ is given by

$$E_\rho = -\lambda \int_G \frac{\partial\theta}{\partial\rho}(g)\,|f(g)|^2, \qquad \text{with } \lambda \text{ a universal (Planck's) constant}$$

Here the term 'energy' is also being used to cover such concepts as momentum, and angular momentum. The interpretation depends on the choice of the group, G, and of the representation (or direction), ρ.

We remarked earlier that G^* can be regarded as 'frequency space'. In order to make numerical, or observational, sense to this term, we must again choose a representation, ρ, from G^* to R^+. This done, the definition below seems reasonable, and correct.

Definition 5.2. The *average* (generalised) *frequency*, ν_ρ, in the direction ρ of a system in a group G with state function $f(g)$ is given by

$$\nu_\rho = \int_{G^*} \rho(g^*)\,|f^*(g^*)|^2$$

If these definitions be accepted, then the generalisation of Planck's assumption arises as the statement of the next, simple theorem.

Theorem 5.1. $E_\rho = \lambda\nu_\rho$.

Proof.

$$\begin{aligned}
\nu_\rho &= \int_{G^*} \rho(g^*)|f^*(g^*)|^2 = \int_G\int_{G^*} \rho(g^*)\bar{f}^*(g^*)f(h)g^*(h^{-1})\\
&= i\int_G \frac{\partial f}{\partial\rho}(h)\bar{f}(h)\\
&= i\int_G \frac{\partial}{\partial\rho}(R(h)e^{i\theta(h)})R(h)e^{-i\theta(h)}\\
&= i\int_G \left(\frac{\partial R}{\partial\rho}\right)(h)\;R(h) + i^2\int_G R^2(h)\frac{\partial\theta}{\partial\rho}(h)\\
&= \frac{i}{2}\int_G \frac{\partial R^2}{\partial\rho}(h) - \int_G |f(h)|^2\frac{\partial\theta}{\partial\rho}(h)\\
&= 0 + E_\rho/\lambda \qquad \text{(by Theorem 3.1(h))}
\end{aligned}$$

This completes the proof.

6. *Conclusions, Suggestions, and Shortcomings*

I would like, finally, to itemise what I believe to be the successes of this approach, together with some of the questions it leaves unanswered. I am no aware of any deficiency which this approach exhibits in relation to the conventional Hilbert space and operators approach. First, the successes:

(1) This theory, offering as the sole condition the minimisation of the integral

$$\int_G \int_{G^*} E(g, g^*) |f(g)|^2 |f^*(g^*)|^2$$

offers an eminently simple criterion for an understanding of the roots of Quantum Mechanics.

(2) The version of the uncertainty relations presented here which reads

$$\int_G |f(g)| \int_{G^*} |f^*(g^*)| \geqslant 1$$

is far easier to explicate than the more familiar mean square formulation, and shows that uncertainty arises out of the character of matter itself, rather than the means of observing it.

(3) For the reader who is irremediably 'hooked' on the mean square formulation, the following is available:

Let $\rho : G \to R^+$ and $\beta : G^* \to R^+$ be continuous homomorphisms from G and G^* respectively to R^+. Let $f(g)$ be a state function on G. With respect to the state function f we define the *mean value*, $\bar{\rho}$, of ρ as

$$\bar{\rho} = \int_G \rho(g) |f(g)|^2$$

Similarly,

$$\bar{\beta} = \int_{G^*} \beta(g) |f^*(g^*)|^2$$

Let

$$\Delta(\rho) = \int_G (\rho(g) - \bar{\rho})^2 |f(g)|^2$$

and

$$\Delta(\beta) = \int_{G^*} (\beta(g) - \bar{\beta})^2 |f^*(g^*)|^2$$

Then

$$\Delta(\rho)\Delta(\beta) \geqslant \frac{1}{4} \left| \frac{\partial \rho}{\partial \beta}(1) \right|$$

This will not be proved here.

(4) We note that uncertainty has been placed in the context of groups, and of symmetry, where it properly belongs, at least so far as physics is concerned. The lessons of Quantum Mechanics are more easily generalised from here than they are from Hilbert spaces, and operator algebras.

Still, this theory does not yet deal adequately with many questions, some of which are presented below.

First, there is the question of selection rules for systems of identical particles. In an entirely satisfactory theory of Quantum Mechanics these rules should arise naturally, and the question of the individuality of particles should arise in some sort of formal uncertainty relation.

Second, not very much has been said here about the problem of measurement, and the associated problem of the effect of Quantum Mechanics on the theory of knowledge, apart from the observation that the failure of particles to have simultaneously sharp position and momentum values is not simply a failure of our means of observation, arising instead as a feature of the particle itself.

Thirdly, it should be mentioned that this method does not deal with mass and velocity as measurable quantities. In defence of this thesis, however, it should be pointed out that these quantities are a classic thorn in the side of Quantum Mechanics.

Finally, there is a very interesting question, unexplored here, of what continuity, in a mathematical sense, has to do with continuity in a physical sense. The possibility presented here, of doing Quantum Mechanics in spaces other than R^n, is that space is, for example, a *discrete* group. The consequences of this, of living in a discrete space which nevertheless appears continuous, are very much in need of examination.

Yet for all this it is still arguable that the Harmonic Analysis presentation of Quantum Mechanics is much to be preferred to the usual presentation because of its increased generality, simplicity, and formal employment of symmetry.

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